Supersingular K3 Surfaces are Unirational

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Lüroth's theorem

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we have

Theorem (Lüroth)

$$L \cong k(u)$$

Unirationality

X an n-dimensional variety over a field k

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Equivalently, there exist inclusions of function fields

$$k \subseteq k(X) \subseteq k(t_1, \ldots, t_n).$$

Curves

Geometric version of Lüroth's theorem:

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Theorem (Lüroth, version 2)

Let X be a smooth and proper curve over a field k. TFAE:

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- X is unirational.
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In fact, we then even have $X \cong \mathbb{P}^1_k$.

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Let X be a smooth and proper surface over an algebraically closed field k **of characteristic zero**.

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Theorem (Zariski)

Let X be a smooth and proper surface over an algebraically closed field k **of positive characteristic**.

TFAE:

- X is birational to $\mathbb{P}^2_k$.
- X is **separably** unirational over k.

Threefolds

Theorem ((Fano), Iskovskikh–Manin, Clemens–Griffiths, Artin–Mumford)

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Unirational surfaces

coming back to surfaces:

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Question When is a surface in positive characteristic unirational?

Necessary conditions for unirationality

Theorem (Shioda)

Let X be a unirational surface. Then,

$$\rho(X) = b_2(X),$$

where

$$\begin{aligned} \rho(X) &= \operatorname{rank} \mathrm{NS}(X) \\ b_2(X) &= \dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}(X, \mathbb{Q}_\ell) \end{aligned}$$

Necessary conditions for unirationality

Definition A surface X is called **Shioda–supersingular** if

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Thus,

$$\text{unirational} \;\Rightarrow\; \text{Shioda–supersingular}$$

Necessary conditions for unirationality

Proposition

Let X be a Shioda–supersingular surface. Then, the F-crystal

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$$\begin{array}{rcl} \text{unirational} & \Rightarrow & \text{Shioda–supersingular} \\ & \Rightarrow & \text{(Artin–)supersingular} \end{array}$$

Converse?

The **Tate–conjecture** gives

$$\text{(Artin–)supersingular} \;\Rightarrow\; \text{Shioda–supersingular}$$

K3 surfaces

Definition A **K3 surface** is a smooth projective surface X over a field k such that

$$\omega_X \cong \mathcal{O}_X \qquad \text{and} \qquad b_1(X) = 0\,.$$

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Example A smooth projective quartic hypersurface

$$X_4 \subset \mathbb{P}^3_k$$

is a K3 surface.

Tate–conjecture

Theorem (Nygaard, Nygaard–Ogus, Maulik, Charles, Madapusi-Pera)

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Corollary For a K3 surface X in odd characteristic,

$$X \text{ is Shioda–supersingular} \quad \Leftrightarrow \quad X \text{ is Artin–supersingular.}$$

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Corollary For a K3 surface X in odd characteristic,

$$X \text{ is Shioda–supersingular} \quad \Leftrightarrow \quad X \text{ is Artin–supersingular.}$$

Thus, for K3 surfaces, only one notion: **supersingular**.

Artin–Rudakov–Shafarevich–Shioda–conjecture

Conjecture (Artin, Rudakov–Shafarevich, Shioda)

For a K3 surface X

$$X \text{ is unirational} \quad \Leftrightarrow \quad X \text{ is supersingular.}$$

Evidence

Theorem

The Artin–Rudakov–Shafarevich–Shioda conjecture holds for

- Shioda-supersingular K3 surfaces in characteristic 2. (Rudakov–Shafarevich)
- Supersingular K3 surfaces with $\sigma_0 \leq 6$ in characteristic 3. (Rudakov–Shafarevich)

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- Supersingular K3 surfaces with $\sigma_0 \leq 3$ in characteristic 5. (Pho–Shimada)

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- Supersingular K3 surfaces with $\sigma_0 \leq 3$ in characteristic 5. (Pho–Shimada)
- Supersingular K3 surfaces with $\sigma_0 \leq 2$ in odd characteristic. (Shioda)

Evidence

Corollary

For every odd prime p, there exists

- a supersingular K3 surface in characteristic p,
- which is unirational.

Formal groups arising from algebraic varieties

X proper over k

Artin and Mazur studied functor

$$\begin{array}{cccc} \Phi^i_{X/k} & : & (\mathrm{Art}/k) & \to & (\text{Abelian groups}) \\ & & S & \mapsto & \ker\left(H^i_{\text{ét}}(X \times_k S, \mathbb{G}_m) \to H^i_{\text{ét}}(X, \mathbb{G}_m)\right) \end{array}$$

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Example (formal Picard group)

If $i = 1$, then

$$\Phi^1_{X/k} \cong \widehat{\mathrm{Pic}}_{X/k}.$$

Formal Brauer group

Fact (Artin–Mazur)

Have tangent-obstruction theory for $\Phi^i_{X/k}$:

tangent space:	$H^i(X, \mathcal{O}_X)$
obstruction space:	$H^{i+1}(X, \mathcal{O}_X)$

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Corollary–Definition

For a K3 surface, $\Phi^2_{X/k}$ is pro-representable by a 1-dimensional formal group law $\widehat{\mathrm{Br}}_{X/k}$ over k,

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Corollary–Definition

For a K3 surface, $\Phi^2_{X/k}$ is pro-representable by a 1-dimensional formal group law $\widehat{\mathrm{Br}}_{X/k}$ over k, called the **formal Brauer group**.

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$G = \mathrm{Spf}\, k[[t]]$ a 1-dimensional formal group law over k,
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$$h(G) := \begin{cases} \infty & \text{if } g(t) = 0 \\ h & \text{if } g(t) = t^{p^h} + \text{higher order terms.} \end{cases}$$

Supersingular K3 surfaces

Fact

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Reminder: want to prove

$$X \text{ supersingular K3} \quad \Rightarrow \quad X \text{ is unirational.}$$

Families of torsors

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Want to classify families of A-torsors

$$\begin{array}{ccccc} X & \supseteq & A & \to & \mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^1 & = & \mathbb{P}^1 & \to & \mathbb{P}^1 \times S \end{array}$$

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- S is spectrum of a local, complete, and Noetherian k-algebra with residue field k.

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- RHS is Cartesian.
- together with a degree N-multisection of $\mathcal{A} \to \mathbb{P}^1 \times S$.

Families of torsors

classified by Abelian group

$$\ker\left(H^1_{\text{ét}}(\mathbb{P}^1 \times S, A) \xrightarrow{\text{res}} H^1_{\text{ét}}(\mathbb{P}^1, A)\right)[N]$$

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Grothendieck–Leray spectral sequence for $X' \xrightarrow{f} \mathbb{P}^1$

$$E_2^{i,j} := H^i_{\text{ét}}(X',\, R^j f_* \mathbb{G}_m) \Rightarrow H^{i+j}_{\text{ét}}(X',\, \mathbb{G}_m)$$

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Then, families of A-torsors over S s.th. there exists degree-N multisection are classified by

$$\widehat{\mathrm{Br}}_X(S)\,[N].$$

Families of torsors

Corollary

Non-trivial families of A-torsors over $S = \operatorname{Spec} k[[t]]$

can and do exist if and only if

- $h(\widehat{\mathrm{Br}}_X) = \infty$,

Families of torsors

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- p divides N.

Families of torsors

Theorem

$X \to \mathbb{P}^1$ supersingular K3 with elliptic fibration with section. Then, there exists a smooth family of supersingular K3 surfaces

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \to & S = \operatorname{Spec} k[[t]] \end{array}$$

- have
$$0 \to \operatorname{Pic}(\mathcal{X}_{\overline{\eta}}) \to \operatorname{Pic}(X) \to \mathbb{Z}/p\mathbb{Z} \to 0$$

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- have inseparable isogenies
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In particular: X unirational $\Leftrightarrow$ $\mathcal{X}_{\overline{\eta}}$ unirational.

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Then, there exists a non-trivial deformation of X

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such that

$$X \text{ unirational} \quad \Leftrightarrow \quad \mathcal{X}_{\overline{\eta}} \text{ unirational}$$

Artin invariant

Proposition–Definition (Artin)

Let X be a supersingular K3 surface in odd characteristic.

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called the **Artin–invariant**.

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Definition

This lattice is called the **supersingular K3 lattice** of Artin invariant σ_0.

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Let X be a supersingular K3 surface in odd characteristic.

- There exists an elliptic fibration $X \to \mathbb{P}^1$, possibly without section.
- If $\sigma_0 \leq 9$, then there exists an elliptic fibration $X \to \mathbb{P}^1$ with section.

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is free $\mathbb{Z}_p$-module of rank 22 with

$$\mathrm{disc}\, T_H = -p^{2\sigma_0}.$$

Moduli of supersingular K3 crystals

Theorem (Ogus)

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- There exists a moduli space $\mathcal{M}_N$ for supersingular K3 crystals $(H, \varphi, \langle -, - \rangle)$ with $N \subset T_H$.
- $\mathcal{M}_N$ is irreducible, smooth and projective over $\mathbb{F}_p$.

Ogus' period map

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$\mathcal{S}_N$ moduli space of supersingular K3 surfaces X with $N \subseteq \mathrm{NS}(X)$.

$\mathcal{M}_N$ moduli space of supersingular K3 crystals H with $N \subset T_H$.

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$\mathcal{S}_N$	moduli space of supersingular K3 surfaces X with $N \subseteq \mathrm{NS}(X)$.
$\mathcal{M}_N$	moduli space of supersingular K3 crystals H with $N \subset T_H$.

Ogus' period map

$$\pi_N : \mathcal{S}_N \to \mathcal{M}_N$$

Ogus' crystalline Torelli theorem

Theorem (Ogus)

In characteristic $p \geq 5$, there exists a period map

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- for X, Y supersingular K3 surfaces

$$X \cong Y \quad \Leftrightarrow \quad H^2_{\mathrm{cris}}(X/W) \cong H^2_{\mathrm{cris}}(Y/W).$$

(RHS: isomorphism of supersingular K3 crystals)

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Corollary

$\mathcal{M}_N$ is an iterated $\mathbb{P}^1$-bundle over $\mathbb{F}_{p^2}$.

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- There does exist a unirational K3 surface (Shioda).

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Supersingular K3 surfaces in characteristic $p \geq 5$ are unirational.